

# $\mathcal{N} = 2$ Supersymmetric Quantum Mechanical Models and Hodge Theory

R. P. Malik<sup>(a,b)</sup> and Avinash Khare<sup>(c)</sup>

<sup>(a)</sup> *Physics Department, BHU-Varanasi-221 005, India*

<sup>(b)</sup> *DST-CIMS, Faculty of Science, BHU-Varanasi-221 005, India*

<sup>(c)</sup> *Indian Institute of Science for Education and Research, Pune 411021, India*

e-mails: malik@bhu.ac.in; khare@iiserpune.ac.in

**Abstract:** We demonstrate the existence of a novel set of discrete symmetries in the context of  $\mathcal{N} = 2$  supersymmetric quantum mechanical (SQM) models in one and two dimensions. We derive the underlying algebra of the continuous symmetry transformations and establish its relevance to the algebraic structures of the de Rham cohomological operators of differential geometry. We further show that the discrete symmetry transformations of our models correspond to the Hodge duality operation so that these models provide interesting examples of Hodge theory.

PACS numbers: 11.30.Pb, 03.65.-w, 02.40.-k

Keywords:  $\mathcal{N} = 2$  supersymmetric quantum mechanics; continuous and discrete symmetries; de Rham cohomological operators; Hodge theory

## 1. Introduction

The supersymmetric (SUSY) quantum mechanical models represent mathematically one of the most beautiful and elegant examples in the realm of theoretical physics which have found applications in diverse domains of physical phenomena (see, e.g. [1,2]). At the level of quantum mechanics, supersymmetry connects the eigenvalues, eigenfunctions as well as scattering matrix of two supersymmetric partner Hamiltonians. On the other hand, at the classical level, this symmetry transforms the commuting dynamical variables into the anticommuting ones and vice-versa. The central theme of our present investigation is to explore a set of continuous and discrete symmetries of totally different kinds of SQM models and demonstrate that these symmetries provide physical realizations of a few abstract properties associated with the de Rham cohomological operators of differential geometry [3-5].

From the physical point of view, the above kind of studies are very important. For instance, in earlier series of research works [6-10], one of us (RPM) has established that, within the framework of Becchi–Rouet–Stora–Tyutin (BRST) formalism, the Abelian 1-form, 2-form and 3-form gauge theories (in 2D, 4D and 6D dimensions of spacetime, respectively) are models for the Hodge theory. Furthermore, the (non-)Abelian 1-form gauge theory in 2D [6], interacting 2D Abelian 1-form theory with Dirac fields [7], 4D free Abelian 2-form gauge theory [8], 6D free Abelian 3-form gauge theory [10], etc., are endowed with continuous and discrete set of symmetry transformations that provide physical realizations for the de Rham cohomological operators, Hodge duality operation, degree of a form, etc., within the purview of BRST formalism. The culmination of all the above studies is the proof that the 2D (non-)Abelian gauge theories (without any interaction with matter fields) belong to a new class of topological field theories [11] and 4D free Abelian 2-form and 6D free Abelian 3-form gauge theories turn out to be the tractable field theoretic models for the quasi-topological field theory [10,12].

All the above cited theoretical models, however, belong to a special class of field theories (i.e. gauge theories) that are endowed with first-class constraints in the terminology of Dirac’s prescription for classification scheme [13]. In a recent paper [14] (hereafter we call it as I), we have taken a one ( $0 + 1$ )-dimensional (1D)  $\mathcal{N} = 2$  SQM model of harmonic oscillator and shown that it provides a physical model for the Hodge theory. The purpose of our present paper is to extend this to a wide class of models. In particular, we consider (i) SQM model with arbitrary superpotential in one dimension (ii) the SQM model in two dimensions corresponding to the motion of a charged particle in an electromagnetic field. In both these cases, we demonstrate that they respect all the pertinent symmetries that provide a basis for these

systems to be the models for the Hodge theory. In particular, we show that all the cohomological operators (and connected Hodge duality operation) find their physical realizations in terms of continuous and discrete symmetry transformations of the theory. Furthermore, the conserved charges, obeying the  $sl(1/1)$  superalgebra, are found to be the analogues of de Rham cohomological operators of differential geometry. In particular, we explicitly discuss the continuous fermionic symmetry transformations (corresponding to  $\mathcal{N} = 2$  supersymmetry) and derive the corresponding supercharges by exploiting the Noether's theorem. We also derive the conserved charge corresponding to a bosonic symmetry that is an anticommutator of the above two SUSY transformations. As expected, it turns out that this bosonic symmetry transformation is equivalent to a time translation.

The plan of the paper is as follows. To set up the notations and conventions, we start off with a brief review of the  $\mathcal{N} = 2$  supersymmetric harmonic oscillator problem discussed in I and discuss its various continuous as well as discrete symmetry transformations in Sec. 2. In Sec. 3, we discuss the continuous symmetries of the two  $\mathcal{N} = 2$  SQM models and derive the corresponding (conserved) Noether charges. Sec. 4 is devoted to a discussion about the discrete symmetries of the two SQM models. We deduce the algebraic structures of the symmetry operators (and the corresponding conserved charges) and establish their connection to the algebra of cohomological operators in Sec. 5. Finally, we make some concluding remarks in Sec. 6.

*Conventions and Notations:* In this paper, the fermionic symmetries [that are the analogue of the nilpotent (co-)exterior derivatives] have been denoted by  $s_1$  and  $s_2$  and their anticommutator (which is an analogue of the Laplacian operator) is denoted by  $s_\omega = \{s_1, s_2\}$  for both the SQM models. The corresponding conserved charges are denoted by  $Q, \bar{Q}, W$ , respectively.

## 2. Preliminaries: SUSY Oscillator

Following I, we begin with the following Lagrangian for a one  $(0 + 1)$ -dimensional (1D) supersymmetric harmonic oscillator which is described by the ordinary bosonic position variable  $x$  and a pair of Grassmannian variables  $(\psi, \bar{\psi})$  (with  $\psi^2 = \bar{\psi}^2 = 0, \psi\bar{\psi} + \bar{\psi}\psi = 0$ ) at the classical level:

$$L_0 = \frac{\dot{x}^2(t)}{2} - \frac{1}{2} \omega^2 x^2(t) + i \bar{\psi}(t) \dot{\psi}(t) - \omega \bar{\psi}(t) \psi(t). \quad (1)$$

Here  $\dot{x} = dx/dt$  and  $\dot{\psi} = d\psi/dt$  are the generalized “velocities” in terms of the variation of the instantaneous bosonic and fermionic variables  $x$  and  $\psi$  with respect to the evolution parameter  $t$ , and for the sake of simplicity, the mass  $m$  of the oscillator has been taken to be one (i.e.  $m = 1$ ).

The above Lagrangian is endowed with the following nilpotent ( $s_1^2 = s_2^2 = 0$ ) fermionic (i.e.  $s_r^2 = 0, s_r \psi + \psi s_r = 0, s_r \bar{\psi} + \bar{\psi} s_r = 0, r = 1, 2$ ) local, continuous and infinitesimal symmetry transformations

$$\begin{aligned} s_1 x &= \frac{-i \psi}{\sqrt{(2 \omega)}}, & s_1 \psi &= 0, & s_1 \bar{\psi} &= \frac{1}{\sqrt{(2 \omega)}} (\dot{x} + i \omega x), \\ s_2 x &= \frac{i \bar{\psi}}{\sqrt{(2 \omega)}}, & s_2 \psi &= 0, & s_2 \bar{\psi} &= \frac{1}{\sqrt{(2 \omega)}} (-\dot{x} + i \omega x), \end{aligned} \quad (2)$$

i.e. the Lagrangian transforms to a total time derivative under  $s_1$  and  $s_2$ . As a consequence, the action integral ( $S = \int dt L_0$ ) remains invariant under the above local, continuous and infinitesimal SUSY transformations.

There is yet another continuous symmetry in the theory that is obtained by taking the anticommutator of the above SUSY transformations  $s_1$  and  $s_2$  modulo a factor of “ $i$ ”. The infinitesimal version of this bosonic symmetry  $s_\omega = \{s_1, s_2\}$  for the relevant dynamical variables of the theory are

$$s_\omega x = \frac{1}{\omega} \dot{x}, \quad s_\omega \psi = \frac{1}{2 \omega} (\dot{\psi} - i \omega \psi), \quad s_\omega \bar{\psi} = \frac{1}{2 \omega} (\dot{\bar{\psi}} + i \omega \bar{\psi}). \quad (3)$$

It can be checked that the Lagrangian in (1) transforms to a total derivative under the above infinitesimal transformations, too, thereby rendering the action integral invariant (see, e.g. I). Thus, we have three continuous symmetries in the theory, out of which, two are fermionic and one is bosonic.

Now we dwell a bit on the existence of a discrete set of symmetries in this model. These transformations are responsible for the beautiful connection between the two SUSY continuous symmetries  $s_1$  and  $s_2$  that have been discussed above. These discrete transformations are

$$x \rightarrow -x, \quad t \rightarrow +t, \quad \omega \rightarrow -\omega, \quad \psi \rightarrow \pm i \bar{\psi}, \quad \bar{\psi} \rightarrow \mp i \psi, \quad (4)$$

under which the Lagrangian (1) remains invariant. Thus, in this model there are, in total, five symmetries, three of them are local, continuous and infinitesimal in nature and two are discrete.

We note that the SUSY symmetry transformation  $s_1$  corresponds to the exterior derivative  $d$  (with  $d^2 = 0$ ) of differential geometry. On the other hand, the nilpotent ( $s_2^2 = 0$ ) SUSY symmetry transformation  $s_2$  stands for the co-exterior derivative  $\delta$  (with  $\delta^2 = 0$ ). This is due to the fact that we have the following operator relationships (see, e.g. [14])

$$s_2 \Phi = \pm * s_1 * \Phi, \quad s_1^2 \Phi = 0, \quad s_2^2 \Phi = 0, \quad \Phi = x, \psi, \bar{\psi}, \quad (5)$$

which mimic the relationship  $\delta = \pm * d *$ ,  $d^2 = \delta^2 = 0$  of differential geometry. It should be noted that the  $*$ , in the above equation (5), corresponds to

the discrete set of symmetries quoted in (4). Thus, the discrete symmetry transformations (4) stand for the Hodge duality  $*$  operation of differential geometry which connects the (co-)exterior derivatives by:  $\delta = \pm * d*$ .

Pertinent to the above discussions, we note that the outcome of two successive discrete transformations on the generic variable  $\Phi(t)$  is

$$* [* \Phi] = + \Phi, \quad \Phi = x, \psi, \bar{\psi}. \quad (6)$$

Following the strictures, laid down by the duality invariant theories [15], there would be only a positive sign in the relationship (5) due to the positive sign present in the above generic equation (6). Thus, the correct version of (5), consistent with a correct duality-invariant theory, is [15]

$$s_2 = + * s_1 *, \quad s_1^2 = 0, \quad s_2^2 = 0, \quad (7)$$

As a consequence, only one of the two transformations, listed in (4), would be physically useful. This can be succinctly expressed as

$$x \rightarrow -x, \quad t \rightarrow +t, \quad \omega \rightarrow -\omega, \quad \psi \rightarrow +i\bar{\psi}, \quad \bar{\psi} \rightarrow -i\psi. \quad (8)$$

To sum up, we have precisely a single discrete symmetry in the theory as given in the above equation. Thus, we conclude that, for the one dimensional theory under consideration, the analogue of the exact relationship between the (co-)exterior derivatives is captured by the relationship  $s_2 = + * s_1 *$ . Dimensionality of our problem also allows the validity of an inverse relationship (i.e.  $s_1 = - * s_2 *$ ) between the SUSY transformations  $s_1$  and  $s_2$ .

We have discussed a bosonic symmetry transformation  $s_\omega = \{s_1, s_2\}$  in the theory that corresponds to the Laplacian operator  $\Delta = (d + \delta)^2 = \{d, \delta\}$ . The operator form of the algebra of the transformations  $s_1, s_2, s_\omega$  matches precisely with the algebra of the de Rham cohomological operators of differential geometry because we have the following exact relationships, namely;

$$\begin{aligned} s_1^2 = 0, \quad s_2^2 = 0, \quad s_\omega = \{s_1, s_2\}, \quad [s_\omega, s_1] = 0, \quad [s_\omega, s_2] = 0, \\ d^2 = 0, \quad \delta^2 = 0, \quad \Delta = \{d, \delta\}, \quad [\Delta, d] = 0, \quad [\Delta, \delta] = 0. \end{aligned} \quad (9)$$

Finally, we have shown, in our earlier work in I, that conserved charges of the 1D SUSY oscillator problem have one-to-one correspondence with the cohomological operators of differential geometry [3-5].

In the next few sections, we demonstrate that the entire discussion presented in this section can be promoted in the cases of (i) the generalizations of oscillator superpotential in one dimension to an arbitrary superpotential, and (ii) the motion of a charged particle in a plane under the influence of an

electromagnetic (EM) field.

### 3. Continuous Symmetries in the Two Models: Conserved Charges

In this section, we take two different kinds of example of  $\mathcal{N} = 2$  supersymmetric quantum mechanical models and discuss their continuous symmetry transformations and derive the corresponding conserved charges by exploiting the fundamental techniques of Noether's theorem. We also establish that these conserved Noether charges are the generators of the above local, continuous and infinitesimal symmetry transformations.

#### 3.1 A Model in One Dimension with Arbitrary Superpotential

We begin with the Lagrangian ( $L_g$ ), which is a generalization of the starting Lagrangian  $L_0$  [cf. (1)], with an arbitrary superpotential  $f(x)$ :

$$L_g = \frac{\dot{x}^2(t)}{2} - \frac{1}{2} \omega^2 (f(x))^2 + i \bar{\psi}(t) \dot{\psi}(t) - \omega f'(x) \bar{\psi}(t) \psi(t), \quad (10)$$

where  $\omega$  is a parameter,  $f(x)$  is an arbitrary real function of  $x$  while  $f'(x) = df/dx$  is the first order derivative on the potential function. Note that in the limit  $f(x) = x$ , we retrieve our original starting one dimensional Lagrangian  $L_0$  for the SUSY harmonic oscillator.

The following nilpotent ( $s_1^2 = 0, s_2^2 = 0$ ) SUSY transformations

$$\begin{aligned} s_1 x &= \frac{-i \psi}{\sqrt{(2 \omega)}}, & s_1 \psi &= 0, & s_1 \bar{\psi} &= \frac{1}{\sqrt{(2 \omega)}} [\dot{x} + i \omega f(x)], \\ s_2 x &= \frac{i \bar{\psi}}{\sqrt{(2 \omega)}}, & s_2 \bar{\psi} &= 0, & s_2 \psi &= \frac{1}{\sqrt{(2 \omega)}} [-\dot{x} + i \omega f(x)], \end{aligned} \quad (11)$$

are the symmetry transformations for the Lagrangian  $L_g$  because

$$s_1 L_g = - \frac{d}{dt} \left[ \frac{\omega f(x) \psi}{\sqrt{(2 \omega)}} \right], \quad s_2 L_g = + \frac{d}{dt} \left[ \frac{i \dot{x} \bar{\psi}}{\sqrt{(2 \omega)}} \right]. \quad (12)$$

As a consequence, the action integral ( $S = \int dt L_g$ ) remains invariant under the above fermionic transformations. Note that the nilpotency properties of  $s_1$  and  $s_2$  are valid only on the on-shell where the following Euler-Lagrange equations of motion

$$\begin{aligned} \ddot{x} + \omega^2 f f' + \omega f'' \bar{\psi} \psi &= 0, & \dot{\psi} + i \omega f' \psi &= 0, & \dot{\bar{\psi}} - i \omega f' \bar{\psi} &= 0, \\ \ddot{\psi} + i \omega f'' \dot{x} \psi + \omega^2 (f')^2 \psi &= 0, & \ddot{\bar{\psi}} - i \omega f'' \dot{x} \bar{\psi} + \omega^2 (f')^2 \bar{\psi} &= 0, \end{aligned} \quad (13)$$

are satisfied. The last two equations, in the above, have been derived from the basic equations of motion  $\dot{\psi} + i\omega f' \psi = 0$ ,  $\dot{\bar{\psi}} - i\omega f' \bar{\psi} = 0$ . Furthermore, it is interesting to check that, under the symmetry transformations  $s_1$  and  $s_2$ , the above equations of motion go to one-another.

There exists a bosonic symmetry  $s_\omega = \{s_1, s_2\}$  in the theory modulo a factor of “ $i$ ”, under which, the physical (dynamical) variables transform as

$$s_\omega x = \frac{1}{\omega} \dot{x}, \quad s_\omega \psi = \frac{1}{2\omega} (\dot{\psi} - i\omega f' \psi), \quad s_\omega \bar{\psi} = \frac{1}{2\omega} (\dot{\bar{\psi}} + i\omega f' \bar{\psi}). \quad (14)$$

Under the above  $s_\omega$  [cf. (14)], the Lagrangian changes as

$$s_\omega L_g = \frac{d}{dt} \left[ \frac{1}{(2\omega)} (\dot{x}^2 - \omega^2 f^2 + i \bar{\psi} \dot{\psi} - \omega f' \bar{\psi} \psi) \right]. \quad (15)$$

As a consequence, the action integral of our present theory remains invariant under the infinitesimal transformations  $s_\omega = \{s_1, s_2\}$ . The above continuous symmetry transformations would lead to the following conserved charges

$$\begin{aligned} Q &= \frac{1}{\sqrt{(2\omega)}} \left[ (-i \dot{x}) + (\omega f(x)) \right] \psi \equiv \frac{1}{\sqrt{(2\omega)}} \left[ (-i p) + (\omega f(x)) \right] \psi, \\ \bar{Q} &= \frac{\bar{\psi}}{\sqrt{(2\omega)}} \left[ (+i \dot{x}) + (\omega f(x)) \right] \equiv \frac{\bar{\psi}}{\sqrt{(2\omega)}} \left[ (+i p) + (\omega f(x)) \right], \\ W &= \frac{1}{\omega} H_g \equiv \frac{1}{\omega} \left[ \frac{p^2}{2} + \frac{\omega^2 f^2}{2} + \omega f' \bar{\psi} \psi \right], \end{aligned} \quad (16)$$

where  $p = (\partial L_g / \partial \dot{x}) = \dot{x}$  is the canonically conjugate momentum w.r.t. the position variable  $x$  and  $H_g$  is the Hamiltonian for the system under consideration. Note that equation of motion  $\dot{\psi} + i\omega f' \psi = 0$  has been used to express  $\dot{\psi}$  in terms of  $\psi$  in the derivation of the Noether charge  $W$ .

### 3.2 Motion of a Charged Particle in Two Dimensions Under the Influence of an EM Field

We consider here the well-known example of the motion of a charged particle in the  $x - y$  plane where the magnetic field ( $B_z$ ) is in the  $z$ -direction. In this case, the Hamiltonian is given by [2,16] ( $\hbar = c = m = e = 1$ )

$$H_{em} = \frac{1}{2} (p_x + A_x)^2 + \frac{1}{2} (p_y + A_y)^2 - B_z \bar{\psi} \psi, \quad (17)$$

where  $A_x(x, y)$ ,  $A_y(x, y)$  are the components of the vector potential in the  $x - y$  plane,  $p_x = \dot{x}$ ,  $p_y = \dot{y}$  are the  $x$  and  $y$  components of the 2D momenta and

$B_z = \partial_x A_y - \partial_y A_x$  is the  $z$ -component of the magnetic field. The Lagrangian for the above system, due to the well-known Legendre transformation, is

$$L_{em} = \frac{1}{2} (\dot{x}^2 + \dot{y}^2) - (\dot{x} A_x + \dot{y} A_y) + i \bar{\psi} \dot{\psi} + B_z \bar{\psi} \psi. \quad (18)$$

The following local, continuous and infinitesimal nilpotent ( $s_1^2 = 0, s_2^2 = 0$ ) fermionic (i.e.  $s_r \psi + \psi s_r, s_r \bar{\psi} + \bar{\psi} s_r = 0, r = 1, 2$ , etc.) transformations

$$\begin{aligned} s_1 x &= \frac{\psi}{\sqrt{2}}, & s_1 y &= \frac{-i \psi}{\sqrt{2}}, & s_1 \psi &= 0, & s_1 \bar{\psi} &= \frac{i}{\sqrt{2}} [\dot{x} - i \dot{y}], \\ s_1 A_x &= \frac{1}{\sqrt{2}} (\partial_x A_x - i \partial_y A_x) \psi, & s_1 A_y &= \frac{1}{\sqrt{2}} (\partial_x A_y - i \partial_y A_y) \psi, \\ s_2 x &= \frac{\bar{\psi}}{\sqrt{2}}, & s_2 y &= \frac{i \bar{\psi}}{\sqrt{2}}, & s_2 \bar{\psi} &= 0, & s_2 \psi &= \frac{i}{\sqrt{2}} [\dot{x} + i \dot{y}], \\ s_2 A_x &= \frac{\bar{\psi}}{\sqrt{2}} (\partial_x A_x + i \partial_y A_x), & s_2 A_y &= \frac{\bar{\psi}}{\sqrt{2}} (\partial_x A_y + i \partial_y A_y), \end{aligned} \quad (19)$$

are the symmetry transformation for the action integral ( $S = \int dt L_{em}$ ) because the Lagrangian (18) transforms to the total derivatives as

$$\begin{aligned} s_1 L_{em} &= - \frac{d}{dt} \left[ \frac{(A_x - i A_y) \psi}{\sqrt{2}} \right], \\ s_2 L_{em} &= + \frac{d}{dt} \left[ \frac{\bar{\psi}}{\sqrt{2}} \{ \dot{x} + i \dot{y} - (A_x + i A_y) \} \right]. \end{aligned} \quad (20)$$

Thus, according to Noether's theorem, we shall have conserved charges which would turn out to be the generators for the above continuous symmetries. These SUSY fermionic ( $Q^2 = \bar{Q}^2 = 0$ ) charges, corresponding to the nilpotent SUSY transformations  $s_1$  and  $s_2$ , are

$$\begin{aligned} Q &= \frac{1}{\sqrt{2}} \left[ (p_x + A_x) - i (p_y + A_y) \right] \psi, \\ \bar{Q} &= \frac{\bar{\psi}}{\sqrt{2}} \left[ (p_x + A_x) + i (p_y + A_y) \right]. \end{aligned} \quad (21)$$

The anticommutator  $\{s_1, s_2\} = s_\omega$  leads to the derivation of a bosonic symmetry in the theory. The local, continuous and infinitesimal version of these transformations (modulo a factor of "i") on the generic variable  $\Phi$  is

$$s_\omega \Phi = \dot{\Phi}, \quad \Phi = x(t), y(t), \psi(t), \bar{\psi}(t), A_x(x, y), A_y(x, y). \quad (22)$$



In the derivation of the above bosonic symmetry transformations, for obvious reasons, we have used the following straightforward inputs, namely;

$$\begin{aligned} \partial_x \psi(t) = 0, \quad \partial_y \psi(t) = 0, \quad \partial_x \bar{\psi}(t) = 0, \quad \partial_y \bar{\psi}(t) = 0, \\ \frac{d}{dt} A_x(x, y) = \dot{x} \partial_x A_x + \dot{y} \partial_y A_x, \quad \frac{d}{dt} A_y(x, y) = \dot{x} \partial_x A_y + \dot{y} \partial_y A_y. \end{aligned} \quad (23)$$

The Lagrangian of our present theory transforms to a total derivative rendering the action integral invariant. The corresponding conserved charge is

$$W = H_{em} \equiv \left[ \frac{(p_x + A_x)^2}{2} + \frac{(p_y + A_y)^2}{2} - B_z \bar{\psi} \psi \right]. \quad (24)$$

Thus, we note that the conserved charge, corresponding to the bosonic symmetry transformations, is nothing but the Hamiltonian of the theory itself.

We wrap up this section with a couple of general remarks. First, the conserved charges in (16), (21) and (24) are the generators of the infinitesimal transformations  $s_1, s_2, s_\omega$  because we have the following relationships

$$s_r \Phi = \mp i [\Phi, Q_r]_{(\pm)}, \quad s_r = s_1, s_2, s_\omega, \quad Q_r = Q, \bar{Q}, W, \quad (25)$$

where the (+)– signs, as the subscripts on the square bracket, stand for the bracket to be (anti)commutator for the generic variable  $\Phi = x, \psi, \bar{\psi}$  and  $\Phi = x, y, \psi, \bar{\psi}, A_x, A_y$  being (fermionic) bosonic in nature for both the  $\mathcal{N} = 2$  SUSY and physically interesting examples of our present endeavor.

## 4. Discrete Symmetries: Duality Transformations

In this section, we discuss the presence of a set of discrete symmetry transformations for both the  $\mathcal{N} = 2$  supersymmetric quantum mechanical models under consideration and establish their relevance to the Hodge duality (\*) operation of differential geometry.

### 4.1 A Model with the Generalized SUSY Potential

It is interesting to note that there is a set of discrete symmetries, under which, the Lagrangian  $L_g$  [cf. (10)] remains invariant:

$$\begin{aligned} x \rightarrow -x, \quad \omega \rightarrow -\omega, \quad \psi \rightarrow \pm i \bar{\psi}, \quad \bar{\psi} \rightarrow \mp i \psi, \\ t \rightarrow t, \quad f(x) \rightarrow -f(x), \quad f'(x) \rightarrow f'(x). \end{aligned} \quad (26)$$

Note that, in the limit  $f(x) = x$ , the transformations (26) go over to the transformations in (4). Further, the fact that the function  $f(x)$  is an odd

function of  $x$  means that, in all these cases, the underlying supersymmetry remains spontaneously *unbroken*.

Following the discussion before Eq. (7) in Sec. 2, ultimately, the following *unique* transformations

$$\begin{aligned} x &\rightarrow -x, & \omega &\rightarrow -\omega, & \psi &\rightarrow +i\bar{\psi}, & \bar{\psi} &\rightarrow -i\psi, \\ t &\rightarrow t, & f(x) &\rightarrow -f(x), & f'(x) &\rightarrow f'(x), \end{aligned} \quad (27)$$

correspond to the Hodge duality  $*$  operation of differential geometry. We dwell a bit now on the importance of these discrete symmetry transformations. The discrete symmetry transformations (27) [that are generalization of (4)] correspond to the Hodge duality  $*$  operation of differential geometry. This can be proven by checking that relations (7) are satisfied by the interplay of continuous and discrete symmetry transformations (11) and (27) when they blend together in a meaningful manner. Furthermore, we find that relation (6) is also valid in the case of general potential function  $f(x)$  for our *first*  $\mathcal{N} = 2$  SUSY example. We observe that relation (9) is also true. Note that the relationship (7) is the analogue of the relationship that exists between the (co-)exterior derivatives  $[(\delta)d]$  of differential geometry.

Actually, there are other possible discrete symmetries in the theory. However, it turns out that none of these symmetries leads to the exact derivation of relationship like (7) and its counterpart  $s_1 = - * s_2 *$ . As a consequence, these discrete symmetries are *not* interesting from the point of view of the duality-invariant physical theories [15]. At this stage, there are a couple of remarks. First, it is interesting that the potential function  $f(x)$  turns out to be odd under parity (see, e.g. [2,16,17] for details). Second, it can be checked that the reverse relationship ( $s_1 = - * s_2 *$ ) of (7) also exists in the theory because of its dimensionality. Henceforth, we shall concentrate on the above unique transformations as the analogue of the Hodge duality  $*$  operation (as far as physical discussions of our present theory, with a general super potential function  $f(x)$ , is concerned).

We close this section with the remark that the super charges  $Q$  and  $\bar{Q}$  transform under the the duality transformations (27) as follows

$$* (Q) = \bar{Q}, \quad * (\bar{Q}) = -Q, \quad * (* Q) = -Q, \quad * (* \bar{Q}) = -\bar{Q}. \quad (28)$$

Note that, in contrast to the transformations (6), the double  $*$  operations on the SUSY (fermionic) charges results in a negative sign. Moreover, the duality  $*$  operation on  $Q$  transforms it to  $+\bar{Q}$  but the same operation on  $\bar{Q}$  takes it back to  $Q$  with an opposite signature.

## 4.2 Motion of a Charged Particle Under Influence of an EM Field

We shall focus here *only* on those discrete symmetry transformations which are useful to us as far as the derivation of connection between SUSY transformations  $s_1$  and  $s_2$  is concerned. In fact, it can be checked that the Lagrangian  $L_{em}$  remains invariant under the following discrete transformations

$$\begin{aligned} x &\rightarrow \mp x, & \psi &\rightarrow \mp \bar{\psi}, & A_x &\rightarrow \pm A_x, & t &\rightarrow -t, \\ y &\rightarrow \pm y, & \bar{\psi} &\rightarrow \pm \psi, & A_y &\rightarrow \mp A_y, & B_z &\rightarrow B_z. \end{aligned} \quad (29)$$

A few remarks are in order at this stage. First, it should be noted that, in reality, there are two discrete transformations in (29) that leave the Lagrangian  $L_{em}$  invariant. Second, there is always a time reversal ( $t \rightarrow -t$ ) symmetry in the theory irrespective of how the coordinates  $(x, y)$ , in the plane, transform. Third, as a consequence of the transformations  $(x \rightarrow \mp x, y \rightarrow \pm y)$ , the space derivatives transform as  $(\partial_x \rightarrow \mp \partial_x, \partial_y \rightarrow \pm \partial_y)$ . Fourth, the kinetic terms  $(\dot{x}^2/2)$  and  $(\dot{y}^2/2)$  remain invariant under (29). Finally, we note that the vector potentials change explicitly, under the parity-type transformations for the space variables  $(x \rightarrow \mp x, y \rightarrow \pm y)$ , as

$$\begin{aligned} A_x(x, y) &\rightarrow A_x(\mp x, \pm y) = \pm A_x(x, y), \\ A_y(x, y) &\rightarrow A_y(\mp x, \pm y) = \mp A_y(x, y), \end{aligned} \quad (30)$$

which leave the magnetic field  $B_z = \partial_x A_y - \partial_y A_x$  invariant.

To appreciate the importance of the discrete symmetry transformations (29), first of all, we observe that two successive operations of these discrete symmetry transformations on any generic variable  $\Phi$  yields either plus or minus sign. This can be mathematically stated as  $*[\Phi] = \pm \Phi$  where the  $*$  operation is nothing but the discrete symmetry transformations (29) and the generic variable  $\Phi = x, y, \psi, \bar{\psi}, A_x, A_y$ . To be more specific, it can be seen that the following is true (for  $\Phi_1$  and  $\Phi_2$  components of  $\Phi$ ) namely;

$$\begin{aligned} *[\Phi_1] &= +\Phi_1, & \Phi_1 &= x, y, A_x, A_y, \\ *[\Phi_2] &= -\Phi_2, & \Phi_2 &= \psi, \bar{\psi}. \end{aligned} \quad (31)$$

The connection between the (co-)exterior derivatives  $(\delta)d$  (i.e.  $\delta = \pm * d *$ ) can be realized between the nilpotent fermionic symmetry transformations  $s_1$  and  $s_2$  (i.e.  $s_2 = \pm * s_1 *$ ). To pin-point this relationship in a more specific fashion (see, e.g. [15]), we have the following explicit relationships, namely;

$$\begin{aligned} s_2 \Phi_1 &= + * s_1 * \Phi_1 \Rightarrow s_2 = + * s_1 *, \\ s_2 \Phi_2 &= - * s_1 * \Phi_2 \Rightarrow s_2 = - * s_1 *, \end{aligned} \quad (32)$$

where, as is evident,  $\Phi_1 = x, y, A_x, A_y$  and  $\Phi_2 = \psi, \bar{\psi}$ . We note that the reverse relationships  $s_1\Phi_1 = - * s_2 * \Phi_1$  and  $s_1\Phi_2 = + * s_2 * \Phi_2$  are also true.

As a final remark, we mention here that the Lagrangian and Hamiltonian of our present model are duality invariant (i.e.  $* L_{em} = L_{em}$ ,  $* H_{em} = H_{em}$ ) and the fermionic conserved charges, transform under (29), as

$$* Q = \mp \bar{Q}, \quad * \bar{Q} = \pm Q, \quad * [* Q] = -Q, \quad * [* \bar{Q}] = -\bar{Q}. \quad (33)$$

Note that, unlike the operations in (31), the double duality  $*$  operations on  $Q$  and  $\bar{Q}$  yield *only* a negative sign.

## 5. Algebraic Structures: Cohomological Aspects

In our present section, we shall discuss the algebraic structures of the conserved charges  $(Q, \bar{Q}, W)$  for both the  $\mathcal{N} = 2$  SUSY quantum mechanical models considered in our present paper.

We have already noted that the fermionic SUSY transformations  $s_1$  and  $s_2$  are nilpotent of order two (i.e.  $s_1^2 = s_2^2 = 0$ ) on the on-shell where the Euler-Lagrange equations of motion (13) are satisfied for our *first* model of SUSY example. In the case of the motion of a charged particle, we observe that the nilpotency of  $s_1$  and  $s_2$  is ensured from the fermionic (i.e.  $\psi^2 = \bar{\psi}^2 = 0$ ) nature of variables  $\psi$  and  $\bar{\psi}$ . Furthermore, it can be explicitly checked that the bosonic symmetry transformation  $s_\omega$ , that is equal to the anticommutator of the fermionic transformations (i.e.  $s_\omega = \{s_1, s_2\}$ ), commutes with both the SUSY transformations  $s_1$  and  $s_2$ . As a consequence, the operator form of the transformations  $s_\omega$  is the Casimir operator for the whole algebra. Thus, we conclude that the operators  $s_1, s_2, s_\omega$  satisfy exactly the same algebra as in the case of SUSY harmonic oscillator [cf. (9)].

It turns out that the conserved charges of (16), (21) and (24) obey exactly the same algebra as the operator form of the transformations  $s_1, s_2, s_\omega$ . Mathematically, this super algebra  $sl(1/1)$  can be succinctly written as

$$Q^2 = 0 \quad \bar{Q}^2 = 0, \quad W = \{Q, \bar{Q}\}, \quad [W, Q] = 0, \quad [W, \bar{Q}] = 0. \quad (34)$$

In view of the fact that  $W = (H_g/\omega)$  and  $W = H_{em}$  for both the SUSY models, respectively, it is obvious that the last two entries in the above equation are nothing but the conservation laws (i.e.  $\dot{Q} = \dot{\bar{Q}} = 0$ ) for  $Q$  and  $\bar{Q}$ . Furthermore, it is clear that the conserved bosonic charge  $W$  is the Casimir operator for the whole algebra. A close look at (34) shows that its algebraic structure is exactly the same as the algebraic structure of the de Rham cohomological operators of differential geometry [(cf. (9))].

Due to the above observations, it is very tempting to identify the set of conserved charges  $(Q, \bar{Q}, W)$  with the set of cohomological operators  $(d, \delta, \Delta)$  of differential geometry. However, the identification is *not* yet complete because the cohomological operators satisfy specific properties when they operate on the differential form of a definite degree. For instance, it is a well-known fact that the (co-)exterior derivatives (lower)raise the degree of a form by *one* when they operate on it. On the contrary, the Laplacian operator does not change the degree of the form on which it acts. We have to capture these properties in the language of conserved charges (i.e.  $Q, \bar{Q}, W$ ) so as to have the complete and correct identification.

To achieve the above goal, in I we had taken the help of the bosonic as well as the fermionic number operators (and their eigen-values). Unfortunately, these arguments fail in the case of arbitrary potential function  $f(x)$ , because the bosonic creation and annihilation operators become non-trivial and their commutation relation produce a first-order derivative  $f'(x)$  on the potential function  $f(x)$ . Similar problem also persists in the case of the second example of our  $\mathcal{N} = 2$  SUSY model of the charge particle.

We now show that, in both cases, this problem can be avoided as follows. Let us first note that the set of operators  $(Q, \bar{Q}, W)$  trivially satisfy:

$$\begin{aligned} [Q \bar{Q}, Q] &= +W Q, & [Q \bar{Q}, \bar{Q}] &= -W \bar{Q}, \\ [\bar{Q} Q, Q] &= -W Q, & [\bar{Q} Q, \bar{Q}] &= +W \bar{Q}, \end{aligned} \quad (35)$$

where, as is evident from (34), the charge  $W = \{Q, \bar{Q}\}$  is the Casimir operator (i.e.  $[W, Q] = [W, \bar{Q}] = 0$ ). We assume that the inverse of the Casimir operator ( $W^{-1}$ ) is well-defined and it certainly commutes with both the nilpotent super charges (i.e.  $[W^{-1}, Q] = [W^{-1}, \bar{Q}] = 0$ ). Hence the algebra (35) can be re-expressed as follows

$$\begin{aligned} \left[ \frac{Q \bar{Q}}{W}, Q \right] &= + Q, & \left[ \frac{Q \bar{Q}}{W}, \bar{Q} \right] &= - \bar{Q}, \\ \left[ \frac{\bar{Q} Q}{W}, Q \right] &= - Q, & \left[ \frac{\bar{Q} Q}{W}, \bar{Q} \right] &= + \bar{Q}, \end{aligned} \quad (36)$$

We now define a state  $|\chi\rangle_p$ , in the quantum Hilbert space of states (QHSS), which satisfies  $(Q\bar{Q}/W) |\chi\rangle_p = p |\chi\rangle_p$  where  $p$  is the eigen-value of operator  $(Q\bar{Q}/W)$ . Using the top two relations of (36), it can be checked that the states  $Q |\chi\rangle_p, \bar{Q} |\chi\rangle_p, W |\chi\rangle_p$  satisfy

$$\begin{aligned} \left( \frac{Q \bar{Q}}{W} \right) Q |\chi\rangle_p &= (p + 1) Q |\chi\rangle_p, \\ \left( \frac{Q \bar{Q}}{W} \right) \bar{Q} |\chi\rangle_p &= (p - 1) \bar{Q} |\chi\rangle_p, \end{aligned}$$

$$\left(\frac{Q\bar{Q}}{W}\right) W |\chi >_p = (p) W |\chi >_p. \quad (37)$$

As a consequence, we note that the states  $Q |\chi >_p, \bar{Q} |\chi >_p, W |\chi >_p$  have the eigen-values  $(p+1), (p-1), (p)$ , respectively. This establishes the fact that if the degree of a form is identified with the eigen-value of a specific state in the QHSS for the operator  $(Q\bar{Q}/W)$ , the result of the operation of conserved charges  $(Q, \bar{Q}, W)$  on this particular state is exactly same as the operation of the cohomological operators  $(d, \delta, \Delta)$  on the specific degree of a form (which is equal to the above eigen-value). Thus, ultimately, we have the following one-to-one mapping

$$(Q, \bar{Q}, W) \quad \Leftrightarrow \quad (d, \delta, \Delta), \quad (38)$$

between the conserved charges corresponding to the physical symmetries of the theory and the cohomological operators of differential geometry.

Now we exploit the lower two relations of (36) and define an arbitrary state  $|\xi >_q$  to possess the eigen-value  $q$  w.r.t. the operator  $(\bar{Q}Q)/W$  [i.e.  $(\bar{Q}Q)/W |\xi >_q = q |\xi >_q$ ]. In view of this definition, we have

$$\begin{aligned} \left(\frac{\bar{Q}Q}{W}\right) Q |\xi >_q &= (q-1) Q |\xi >_q, \\ \left(\frac{\bar{Q}Q}{W}\right) \bar{Q} |\xi >_q &= (q+1) \bar{Q} |\xi >_q, \\ \left(\frac{\bar{Q}Q}{W}\right) W |\xi >_q &= (q) W |\xi >_q. \end{aligned} \quad (39)$$

The above relationship establishes that the states  $Q |\xi >_q, \bar{Q} |\xi >_q, W |\xi >_q$  have the eigen-values  $(q-1), (q+1), (q)$ , respectively. Thus, we conclude that if the degree of a form is identified with the eigen-value  $q$  of a state in the QHSS corresponding to the operator  $(\bar{Q}Q)/W$ , then, there is one-to-one relationship between the conserved charges  $(\bar{Q}, Q, W)$  corresponding to the continuous symmetries of the theory and the cohomological operators:

$$(\bar{Q}, Q, W) \quad \Leftrightarrow \quad (d, \delta, \Delta). \quad (40)$$

We have thus proven that both of our  $\mathcal{N} = 2$  SUSY models are interesting physical models for the Hodge theory where all the de Rham cohomological operators, Hodge duality operation, degree of a form, etc., find their physical realizations in the language of discrete and continuous symmetry transformations (and corresponding generators as conserved charges).

## 6. Conclusions

In our present investigation, we have shown that two well-known  $\mathcal{N} = 2$  SUSY quantum mechanical systems, in one and two dimensions, are tractable models for the Hodge theory. In view of the generality of the arguments, we conjecture that any arbitrary  $\mathcal{N} = 2$  SUSY quantum mechanical model would also be a tractable model for the Hodge theory.

It is amusing to note one interesting distinction between the one and the two dimensional models discussed by us. In particular, for the 1D system of  $\mathcal{N} = 2$  SUSY model, we obtain only one discrete symmetry transformation that is consistent with the strictures laid down by the duality-invariant physical theories [15]. As a consequence, we have only one relationship between the SUSY transformations  $s_1$  and  $s_2$  (i.e.  $s_2 = + * s_1 *$ ) as an analogue of the well-known connection between the (co-)exterior derivatives:  $\delta = \pm * d *$ . On the contrary, for the 2D case of the motion of a charged particle (corresponding to our *second* example of  $\mathcal{N} = 2$  SUSY model), we have two discrete symmetries [cf. (29)]. As a result, we have two relationships  $s_2 = \pm * s_1 *$  that are precise analogues of the relationships between the (co-)exterior derivatives:  $\delta = \pm * d *$  of differential geometry.

It is an open question as to why there is only one physically consistent discrete symmetry for the 1D  $\mathcal{N} = 2$  SUSY model whereas there are, physically consistent, two discrete symmetry transformations for the 2D model of  $\mathcal{N} = 2$  SUSY system.

**Acknowledgement:** One of us (RPM) is grateful to the Director, IISER, Pune, for the warm hospitality extended to him during his visit to the THEP group (of IISER) where a part of this work was completed.

## References

- [1] See, e.g., E. Witten, Nucl. Phys. B **185**, 531 (1981).
- [2] See, e.g., for an excellent review, F. Cooper, A. Khare and U. Sukhatme, Phys. Rep. **251**, 264 (1995).
- [3] T. Eguchi, P. B. Gilkey and A. Hanson, Phys. Rep. **66**, 213 (1980).

- [4] See, e.g., S. Mukhi and N. Mukanda, *Introduction to Topology, Differential Geometry and Group Theory for Physicists* (Wiley Eastern Private Limited, New Delhi, 1990).
- [5] K. Nishijima, Prog. Theor. Phys. **80**, 897 (1988).
- [6] R. P. Malik, Int. J. Mod. Phys. A **22**, 3521 (2007).
- [7] R. P. Malik, Mod. Phys. Lett. A **15**, 2079 (2000), *ibid.* A **16**, 477 (2001).
- [8] S. Gupta and R. P. Malik, Eur. Phys. J. C **58**, 517 (2008).
- [9] R. Kumar, S. Krishna, A. Shukla and R. P. Malik, Eur. Phys. J. C **72**, 1980 (2012).
- [10] R. Kumar, S. Krishna, A. Shukla and R. P. Malik, arXiv:1203.5519 [hep-th].
- [11] R. P. Malik, J. Phys. A: Math. Gen. **41**, 4167 (2001).
- [12] R. P. Malik, J. Phys. A: Math. Gen. **36**, 5095 (2003).
- [13] P. A. M. Dirac, *Lectures on Quantum Mechanics* [Belfer Graduate School of Science] (Yeshiva University Press, New York, 1964).
- [14] R. Kumar and R. P. Malik, Euro. Phys. Lett. **98**, 11002 (2012).
- [15] S. Deser, A. Gomberoff, M. Henneaux and C. Teitelboim, Phys. Lett. B **400**, 80 (1997).
- [16] A. Khare and J. Maharana, Nucl. Phys. B **244**, 409 (1984).
- [17] See, e.g., for an excellent exposition, A. Das, *Field Theory: A Path Integral Approach* (World Scientific, Singapore, 1993).